

Feynman parametrization.

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Feynman's denominator formula, or simply Feynman parametrization, is quite useful and a well known result in high energy physics for evaluating certain types of Feynman diagrams, but I have not explicitly found (even though it of course might be out there on the internet somewhere) a detailed proof of this formula, so here is my attempt at such a detailed proof¹. This is as well an exercise for me to make this derivation, and I hope you will find it useful:

Theorem 1 (Feynman parametrization or Feynman's denominator formula).

$$\frac{1}{\prod_{i=1}^n A_i^{\alpha_i}} = \frac{\Gamma(\sum_{i=1}^n \alpha_i)}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^1 dx_1 \dots dx_n \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{(\prod_{i=1}^n x_i^{\alpha_i-1})}{(\sum_{i=1}^n A_i x_i)^{(\sum_{i=1}^n \alpha_i)}}$$

Proof. First we note that the definition of the Euler- Γ function is:

$$\Gamma(\alpha) = \int_0^\infty dt (t^{\alpha-1} e^{-t})$$

and that for the case where $\alpha \in \mathbb{N}$, we have that:

$$\Gamma(\alpha) = (\alpha - 1)!$$

It is not hard to generalize this formula a tiny amount to make another family of functions:

$$\Gamma_A(\alpha) = \int_0^\infty dt (t^{\alpha-1} e^{-At}) = \frac{\Gamma(\alpha)}{A^\alpha}$$

Now, imagine that we want the product between n of such functions, all with different A_i and α_i , where the subscript i is introduced only to distinguish between different A_i and α_i .

$$\prod_{i=1}^n \Gamma_{A_i}(\alpha_i) = \prod_{i=1}^n \int_0^\infty dt_i (t_i^{\alpha_i-1} e^{-A_i t_i}) = \frac{\prod_{i=1}^n \Gamma(\alpha_i)}{\prod_{i=1}^n A_i^{\alpha_i}}$$

It is important to note here as well that this identity is not necessarily true for infinite values of n , thus we restrict ourselves to the case of finite n . To proceed from here, let the sum of all the integration variable be s , such that:

$$s = \sum_{i=1}^n t_i$$

To clarify, the ranges of each of the $t_i = [0, \infty]$, $\forall i \in \{1, 2, \dots, n\}$, and thus we have that $s = [0, \infty]$ we can expand the integral part of the previous equation, by using a Dirac- δ function such that:

$$\prod_{i=1}^n \Gamma_{A_i}(\alpha_i) = \int_0^\infty ds \left(\prod_{i=1}^n \int_0^\infty dt_i (t_i^{\alpha_i-1} e^{-A_i t_i}) \right) \delta\left(s - \sum_{i=1}^n t_i\right) = \frac{\prod_{i=1}^n \Gamma(\alpha_i)}{\prod_{i=1}^n A_i^{\alpha_i}}$$

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¹It should be noted though, that indirect proofs are plentiful and some quite good as well. Look no further than the wikipedia page for Feynman parametrization, and you will see derivations of some simplified cases, but not the full machinery.

From here, new normalized variables x_i are introduced

$$x_i = \frac{t_i}{s} \quad , \quad x_i \in [0, 1]$$

We can now make the following substitution²

$$\prod_{i=1}^n \Gamma_{A_i}(\alpha_i) = \int_0^\infty ds \left(\prod_{i=1}^n \int_0^1 dx_i s ((sx_i)^{\alpha_i-1} e^{-A_i s x_i}) \right) \delta \left(s \left(1 - \sum_{i=1}^n x_i \right) \right)$$

and using the delta function identity that:

$$\delta \left(s \left(1 - \sum_{i=1}^n x_i \right) \right) = \frac{1}{s} \delta \left(1 - \sum_{i=1}^n x_i \right)$$

we can write the above equation as:

$$\begin{aligned} \prod_{i=1}^n \Gamma_{A_i}(\alpha_i) &= \int_0^\infty ds \left(\prod_{i=1}^n \int_0^1 dx_i s ((sx_i)^{\alpha_i-1} e^{-A_i s x_i}) \right) \frac{1}{s} \delta \left(1 - \sum_{i=1}^n x_i \right) \\ &= \int_0^\infty ds \left(\int_0^1 dx_1 \dots dx_n s^n \left(\prod_{i=1}^n s^{\alpha_i-1} x_i^{\alpha_i-1} e^{-A_i s x_i} \right) \right) \frac{1}{s} \delta \left(1 - \sum_{i=1}^n x_i \right) \\ &= \int_0^1 dx_1 \dots dx_n \left(\prod_{i=1}^n x_i^{\alpha_i-1} \right) \delta \left(1 - \sum_{i=1}^n x_i \right) \int_0^\infty ds \left(\frac{1}{s} s^n s^{\sum_{i=1}^n (\alpha_i-1)} e^{-s \sum_{i=1}^n (A_i x_i)} \right) \\ &= \int_0^1 dx_1 \dots dx_n \left(\prod_{i=1}^n x_i^{\alpha_i-1} \right) \delta \left(1 - \sum_{i=1}^n x_i \right) \int_0^\infty ds \left(s^{\sum_{i=1}^n (\alpha_i)-1} e^{-s \sum_{i=1}^n (A_i x_i)} \right) \end{aligned}$$

and noting that the s -integral has the exact form of the $\Gamma_A(\alpha)$ function, we can make the identification,

$$\int_0^\infty ds \left(s^{\sum_{i=1}^n (\alpha_i)-1} e^{-s \sum_{i=1}^n (A_i x_i)} \right) = \Gamma_{\sum_{i=1}^n A_i x_i} \left(\sum_{i=1}^n \alpha_i \right) = \frac{\Gamma(\sum_{i=1}^n \alpha_i)}{(\sum_{i=1}^n A_i x_i)^{(\sum_{i=1}^n \alpha_i)}}$$

and thus:

$$\begin{aligned} \prod_{i=1}^n \Gamma_{A_i}(\alpha_i) &= \int_0^1 dx_1 \dots dx_n \left(\prod_{i=1}^n x_i^{\alpha_i-1} \right) \delta \left(1 - \sum_{i=1}^n x_i \right) \int_0^\infty ds \left(s^{\sum_{i=1}^n (\alpha_i)-1} e^{-s \sum_{i=1}^n (A_i x_i)} \right) \\ &= \int_0^1 dx_1 \dots dx_n \left(\prod_{i=1}^n x_i^{\alpha_i-1} \right) \delta \left(1 - \sum_{i=1}^n x_i \right) \frac{\Gamma(\sum_{i=1}^n \alpha_i)}{(\sum_{i=1}^n A_i x_i)^{(\sum_{i=1}^n \alpha_i)}} \\ &= \Gamma \left(\sum_{i=1}^n \alpha_i \right) \int_0^1 dx_1 \dots dx_n \delta \left(1 - \sum_{i=1}^n x_i \right) \frac{(\prod_{i=1}^n x_i^{\alpha_i-1})}{(\sum_{i=1}^n A_i x_i)^{(\sum_{i=1}^n \alpha_i)}} \end{aligned}$$

since the $\Gamma(\sum_{i=1}^n \alpha_i)$ factor is independent of the x_i . Now, using what we have found, remember what the goal was; to find an integral-parametrized form of $\frac{1}{\prod_{i=1}^n A_i^{\alpha_i}}$, we can also remember that the initial representation we found for the $\Gamma_A(\alpha)$ function, we have actually (almost) arrived at the end result:

$$\prod_{i=1}^n \Gamma_{A_i}(\alpha_i) = \frac{\prod_{i=1}^n \Gamma(\alpha_i)}{\prod_{i=1}^n A_i^{\alpha_i}} = \Gamma \left(\sum_{i=1}^n \alpha_i \right) \int_0^1 dx_1 \dots dx_n \delta \left(1 - \sum_{i=1}^n x_i \right) \frac{(\prod_{i=1}^n x_i^{\alpha_i-1})}{(\sum_{i=1}^n A_i x_i)^{(\sum_{i=1}^n \alpha_i)}}$$

²I will ask you to forgive my notation; when I write $\int_0^1 dx_1 \dots dx_n$, I implicitly refer to the nested integrals $\int_0^1 dx_1 \dots \int_0^1 dx_n$ it is simply a notation that in the attempt of minimizing the clutter in writing, abuse the fact that $x_i \in [0, 1]$, $\forall i \in \{1, \dots, n\}$

whereby we can conclude with the formula:

$$\frac{1}{\prod_{i=1}^n A_i^{\alpha_i}} = \frac{\Gamma(\sum_{i=1}^n \alpha_i)}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^1 dx_1 \dots dx_n \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{(\prod_{i=1}^n x_i^{\alpha_i - 1})}{(\sum_{i=1}^n A_i x_i)^{(\sum_{i=1}^n \alpha_i)}}$$

□